It is as follows: Take n solids of revolution corresponding to the volumes x, x^2, $x^3 \ldots . x^n$ and attach them to a lever which may turn about a fixed edge H. The axes of these solids must be kept in a vertical position. Let a_1, a_2, $a_3 \ldots . a_n$ be the distances from H of the points where these axes intersect the lever. Attach also a weight 1 to the lever at a distance $-a_0$ from H. According as a_k is positive or negative attach the corresponding solid to the right or left of H. The lowest points of the solids are all in the same straight line. The whole apparatus is now put in a vessel containing water and the level of water is raised until the lever and with it the lowest points of the solids are all in a horizontal position. In this position the depths of immersion of the solids are all equal to a certain length y and the quantities of water displaced are y, y^2, $y^3 \ldots .$ y^n. These act upon the lever as vertical forces in an upward direction.

The moment of all these forces is therefore

$$a_0+a_1y+a_2y^2+\ldots .+a_ny^n \ldots .(2).$$

The other forces acting upon the lever are the weights of the solids themselves. If these are all equal to w, their moment is

$$w(a_1+a_2+a_3+\ldots .+a_n), \text{ hence}$$

$$a_0+a_1y+a_2y^2+\ldots .+a_ny^n+w(a_1+a_2+a_3+\ldots .+a_n)=0 \ldots \quad (3).$$

Adding now a weight w at a distance $-(a_1+a_2+a_3+\ldots .+a_n)$ from H and adjusting the water-level in such a manner that the lever assumes again a horizontal position, then the depth of immersion will be x, since in this case

$$a_0+a_1x+a_2x^2+\ldots .+a_nx^n=0 \ldots .(4).$$

Hence, the depth of immersion of the solids is a positive real root of (1).

After the first root has been found, the water-level in the vessel may be raised still higher. Equilibrium will first be destroyed, but at a certain point of the level, equilibrium is again restored and the corresponding depth of immersion will give another positive real root of (1). By this process all positive real roots of the proposed equation may be obtained.

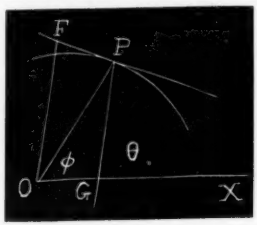
I shall not enter into further details of this interesting problem. Since the publication of my first article on this subject I have found that M. Georges Meslin has published a similar method in the *Journal de Physique* for June, 1900, Vol. IX, III series, page 339. The reader is referred to Professor Meslin's valuable article, where a description of an apparatus for the solution of equations (1) may be found.

Boulder, Col., February 9, 1901.

THE LENGTH OF A DEGREE OF LATITUDE AND LONGITUDE FOR ANY PLACE.

By GEORGE B. McCLELLAN ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let O be the center of the earth, P any place on the surface, PG the normal and PF the tangent at P, $OP=p$=the perpendicular from the center on the tangent. a, b the semi-axes of the earth, ρ=radius of curvature at P, $OP=r$ =the radius vector, $\angle POX=\varphi$, $\angle PGX=FOX=\theta$, l=length of a degree of meridian, ρ'=radius of circle of latitude, L=length of degree of latitude, $e^2=\frac{a^2-b^2}{a^2}$=.00680349=square of the eccentricity, since a=6378190 meters=3963.296 miles.



Now $\rho=\frac{a^2b^2}{p^3}$, but $p=\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}=a\sqrt{1-e^2\sin^2\theta}$.

$$\therefore\ \rho=\frac{a(1-e^2)}{(1-e^2\sin^2\theta)^{\frac{3}{2}}},\quad l=\frac{2\pi\rho}{360}=\frac{\pi\rho}{180}.$$

$$\therefore\ l=\frac{\pi a(1-e^2)}{180(1-e^2\sin^2\theta)^{\frac{3}{2}}}=\frac{68.70175}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}\text{ miles}=\frac{110562.7346}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}\text{ meters.}$$

$$\frac{1}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}=1+\tfrac{3}{2}e^2\sin^2\theta+\tfrac{15}{8}e^4\sin^4\theta+\tfrac{105}{48}e^6\sin^6\theta+\ldots.$$

The fourth term can be omitted as its greatest value will not affect the result more than three inches.

$$\sin^2\theta=\tfrac{1}{2}(1-\cos2\theta),\quad \sin^4\theta=\tfrac{1}{8}(3-4\cos2\theta+\cos4\theta).$$

These values, with the value of e, give

$$\frac{1}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}=1.00514058-.00515323\cos2\theta+.00001265\cos4\theta.$$

$$\therefore\ l=69.054917-.354036\cos2\theta+.000869\cos4\theta\text{ miles}$$

$$=111131.09118-569.75520\cos2\theta+1.39862\cos4\theta\text{ meters.}$$

$$\rho'=r\cos\varphi=\frac{a\cos\theta}{\sqrt{(1-e^2\sin^2\theta)}}.$$

$$L=\frac{2\pi\rho'}{360}=\frac{\pi\rho'}{180}=\frac{\pi a\cos\theta}{180\sqrt{(1-e^2\sin^2\theta)}}.$$

$$\therefore L=\frac{69.1726\cos\theta}{\sqrt{(1-e^2\sin^2\theta)}}\text{ miles}=\frac{111320.4635\cos\theta}{\sqrt{(1-e^2\sin^2\theta)}}\text{ meters.}$$

$$\frac{1}{\sqrt{(1-e^2\sin^2\theta)}}=1+\tfrac{1}{2}e^2\sin^2\theta+\tfrac{3}{8}e^4\sin^4\theta+\tfrac{1}{4}\tfrac{5}{8}e^6\sin^6\theta+\ldots.$$

$$=(1+\tfrac{1}{2}e^2+\tfrac{3}{8}e^4)-(\tfrac{1}{2}e^2+\tfrac{3}{4}e^4)\cos^2\theta+\tfrac{3}{8}e^4\cos^4\theta\ldots.$$

$$\cos^3\theta=\tfrac{1}{4}(\cos3\theta+3\cos\theta),\quad \cos^5\theta=\tfrac{1}{16}(\cos5\theta+5\cos3\theta+10\cos\theta).$$

$$\therefore L=111320.4635[(1+\tfrac{1}{8}e^2+\tfrac{3}{64}e^4)\cos\theta-(\tfrac{1}{8}e^2+\tfrac{9}{128}e^4)\cos3\theta+\tfrac{9}{128}e^4\cos5\theta]$$

$$=111415.37533\cos\theta-95\ 03428\cos3\theta+.12022\cos5\theta.$$

Since the greatest value of the last term is not over five inches it can be omitted.

$$\therefore L=111415.37533\cos\theta-95.03428\cos3\theta\text{ meters}$$

$$=69.23155\cos\theta-.05905\cos3\theta\text{ miles.}$$

The following table gives the length of a degree at intervals of five degrees.

	⌐——*l*——⌐		⌐——*L*——⌐	
Degrees	*Meters*	*Miles*	*Meters*	*Miles*
0°	110562.7346	68.70175	111320.3411	69.17250
5°	110571.305	68.70701	111051.725	69.00558
10°	110596.769	68.72288	109640.673	68.12878
15°	110638.365	68.74873	107552.254	66.83108
20°	110694.879	69.78386	104648.397	65.02667
25°	110764.615	68.82719	100952.272	62-72996
30°	110845.514	68.87746	96489.058	59.95660
35°	110935.152	68.93315	91290.501	56.72631
40°	111030.839	68.99261	85396.151	53.06366
45°	111129.693	69.05405	78850.126	48.99608
50°	111228.715	69.11556	71698.992	44.55249
55°	111324.887	69.17532	63997.427	39.96687
60°	111415.270	69.23150	55802.722	34.67483
65°	111497.081	69.28234	47178.162	29.31568
70°	111567.789	69.32627	38188.589	23.72972
75°	111625.216	69.36194	28903.727	17.96027
80°	111667.556	69.38825	19394.797	12.05159
85°	111693.506	69.40438	9735.561	6.04951
90°	111702.245	69.40988	0.000	0.000

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

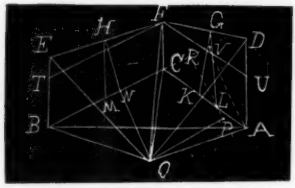
ARITHMETIC.

137. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

At the corners of a triangle sides a, b, c feet, are towers d, e, f feet high. At what point must a ladder be placed so that it will just reach to the top of each tower without moving? How long is the ladder? Substitute $a=200$, $b=180$, $c=150$, $d=60$, $e=50$, $f=30$; d, e, f at A, B, C, respectively.

Solution by the PROPOSER.

Let ABC be the triangle, $AD=d$, $BE=e$, $CF=f$, the towers. Join DF, EF, and draw UF parallel to AC, TF parallel to BC. From G, the mid-point of DF, draw GK parallel to AD, GL perpendicular to DF. From H, the mid-point of EF, draw HM parallel to BE, HN parallel to EF. Draw ON perpendicular to BC, and OL perpendicular to AC. Then O is the required foot of the ladder. For O is equally distant from D, E, F, since OL is perpendicular to the plane $ADFC$ at L, and ON is perpendicular to the plane $BCFE$ at N. Draw LR, AV perpendicular to BC, OP perpendicular to LR.



Then $DU=d-f$, $EF=e-f$, $GK=\frac{1}{2}(d+f)$, $HM=\frac{1}{2}(e+f)$.

In the similar triangles LGK and DFU, $LK:GK=DU:UF$.

$$\therefore\ LK=\frac{d^2-f^2}{2b},\quad CL=\tfrac{1}{2}b+\frac{d^2-f^2}{2b}=\frac{b^2+d^2-f^2}{2b}.$$

Similarly $MN=\dfrac{e^2-f^2}{2a}$, $CN=\tfrac{1}{2}a-\dfrac{e^2-f^2}{2a}=\dfrac{a^2+f^2-e^2}{2a}$.

$AV=\dfrac{2\triangle}{a}$, where $\triangle$ = area ABC. $VC=\sqrt{\dfrac{b^2-4\triangle^2}{a^2}}$.

$$\therefore\ VC=\frac{a^2+b^2-c^2}{2a}.\quad RC:LC=VC:AC.$$

$$\therefore\ RC=\frac{(b^2+d^2-f^2)(a^2+b^2-c^2)}{4ab^2}.$$

$$RN=OP=RC+NC=\frac{(b^2+d^2-f^2)(a^2+b^2-c^2)+2b^2(a^2+f^2-e^2)}{4ab^2}.$$

$OL:OP=AC:AV$.

$$\therefore OL=\frac{(b^2+d^2-f^2)(a^2+b^2-c^2)+2b^2(a^2+f^2-e^2)}{8\triangle b}.$$

$OF=\sqrt{(OL^2+CL^2+CF^2)}$.
When $a=200$, $b=180$, $c=150$, $d=60$, $e=50$, $f=30$,
$OL^2=60363.9509$, $CL^2=9506.25$, $CF^2=900$.
$\therefore OF=266.03$ feet.

Also solved by *J. SCHEFFER.*

136. Proposed by F. M. PRIEST, Mona House, St. Louis, Mo.

"A pound of gold may be drawn into a wire that would extend around the earth." What would be the diameter of such a wire if the specific gravity of gold is 19.36 and the distance is 24,900 miles?

Solution by J. M. ARNOLD, Crompton, R. I.; G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.; and J. SCHEFFER, A. M., Hagerstown, Md.

62.4 pounds=weight of 1 cubic foot of water. Then, the specific gravity of gold being 19.36, the weight of 1 cubic foot of gold is 19.36×62.4 pounds, or 1208.064 pounds.

Hence, in 1 pound of gold there are $\frac{1728}{1208.64}$ or 1.43039 cu. inches nearly.

$\therefore \frac{1}{4}\pi d^2\times24900\times5280\times12=1.43039$ cubic inches.

$\therefore d=.000034$ inches, nearly.

Mr. Arnold remarks that to measure so small a quantity one would have to estimate 1-12 of one of the divisions of a Brown and Sharp's Micrometer Gage, which reads to the hundredth of a millimeter.

Also solved by *ELMER SCHUYLER.*

139. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College Mechanicsburg, Pa.

The *ratio* of the interest to the true discount on a certain principal for a certain time at a certain rate per cent. per annum, is $m=21$ to $n=20$. What is the rate per cent.?

Solution by P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.; and ELMER SCHUYLER, M. Sc., Professor of Mathematics, Boys' High School, Reading, Pa.

Let P be the principal; r, the rate; and t, the time in years.
Then the interest, I, is trP.

The true discount$=\frac{trP}{1+rt}$. $\therefore trP:\frac{trP}{1+tr}=m:n=21:20$.

$\therefore 1+rt=m/n=\frac{21}{20}$, and $rt=\frac{m-n}{n}=\frac{1}{20}$, or $r=\frac{1}{20t}$.

Thus r depends on the time.
If $t=1$ year, $r=5\%$.

Also solved by *G. B. M. ZERR, J. M. ARNOLD.* and *J. SCHEFFER.*

ALGEBRA.

113. Proposd by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find a square number consisting of 24 figures such that the numbers formed by the first 12 figures and the last 12 figures, respectively, are consecutive, and *vice versa*.

Solution by the PROPOSER.

(1). Let $y^2=1000000000001x+1000000000000$ be the number.

$\therefore 1000000000001x=y^2-1000000000000.$

$$\therefore x=\frac{(y+1000000)(y-1000000)}{10001\times 99990001}.$$

Let $y+1000000=99990001u$, $y-1000000=10001v$.

$$\therefore 10001v=99990001u-2000000. \quad v=9998u-199+\frac{3u-9801}{10001}.$$

$$\text{Let } S=\frac{3u-9801}{10001}. \quad \therefore u=\frac{10001S+9801}{3}=3333S+3267+\frac{2S}{3}.$$

Let $S=3m$. $\therefore u=10001m+3267.$

$y=1000000000001m+326666333267.$

$x=(99990001m+32663267)(10001m+3267).$

Let $m=0$. $\therefore y=326666333267$, $x=106710893289$.

$y^2=10671089329010671089328 9=1000000000000(x+1)+x.$

(2). $y^2=1000000000001x+1$, or $x=\dfrac{(y+1)(y-1)}{(10001)(99990001)}$.

$y+1=99990001u$, $y-1=10001v$.

$$\therefore v=\frac{9998u+(3u-2)}{10001}=9998u+S. \quad \therefore u=\frac{3333S+(2S+2)}{3}=3333S+t.$$

$\therefore S=t-1+\frac{1}{3}t=t-1+m. \quad \therefore S=3m-1, \ u=10001m-3333.$

$y=1000000000001m-333266673334.$

$x=(10001m-3333)(99990001m-33323335).$

Let $m=1$.

$\therefore y=666733326667$, $x=444533328888$.

$y^2=444533328888444533328889=1000000000000x+(x+1).$

114. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics in Boys' High School, Reading, Pa.

$$\left.\begin{aligned} bx^3&=10a^2bx+3a^3y \\ ay^3&=10b^2ay+3b^3x \end{aligned}\right\}$$

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and the PROPOSER.

$bx^3=10a^2bx+3a^3y$.....(1), $ay^3=10ab^2y+3b^3x$.....(2).

Let $x/a=u$, $y/b=v$, then (1) and (2) become

$u^3=10u+3v$.....(3), $v^3=10v+u$.....(4).

(3)+(4) gives $u^3+v^3-13(u+v)=0$.....(5).

(3)−(4) gives $u^3-v^3-7(u-v)=0$.....(6).

From (5), $u+v=0$..... (7); $u^2-uv+v^2=13$..... (8).

From (6), $u-v=0$.....(9), $u^2+uv+v^2=7$.....(10).

From (7) and (9), $u=0$, $v=0$.

(8)+(10) gives $u^2+v^2=10$ .....(11).

(8)−(10) gives $uv=-3$.....(12).

From (11) and (12), $u+v=\pm 2$, $u-v=\pm 4$.

$\therefore u=\pm 3$ or ± 1, $v\mp 1$ or ∓ 3.

From (7) and (10), $u=\pm\sqrt{7}$, $v=\mp\sqrt{7}$.

From (9) and (8), $u=\pm\sqrt{13}$, $v=\pm\sqrt{13}$.

$\therefore u=0,\ 1,\ -1,\ 3,\ -3,\ \sqrt{7},\ -\sqrt{7},\ \sqrt{13},\ -\sqrt{13}$.

$v=0,\ -3,\ 3,\ -1,\ 1,\ -\sqrt{7},\ \sqrt{7},\ \sqrt{13},\ -\sqrt{13}$.

$x=0,\ a,\ -a,\ 3a,\ -3a,\ a\sqrt{7},\ -a\sqrt{7},\ a\sqrt{13},\ -a\sqrt{13}$.

$y=0,\ -3b,\ 3b,\ -b,\ b,\ -b\sqrt{7},\ b\sqrt{7},\ b\sqrt{13},\ -b\sqrt{13}$.

II. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Let $bx/ay=t$.

Divide (1) by (2), then $\dfrac{b^3x^3}{a^3y^3}=\dfrac{10bx+3ay}{10ay+3bx}$, or $t^3=\dfrac{10t+3}{10+3t}$.

Whence, $3t^4+10t^3-10t-3=0$.

Factoring, $(t^2-1)(3t^2+10t+3)=0$.

Whence, $t=1,\ -1,\ -\frac{1}{3},\ -3$.

When $t=1$, $bx=ay$ and substituting in (1) we easily get,

$x=0$, or $\pm a\sqrt{13}$.
$y=0$, or $\pm b\sqrt{13}$.

When $t=-1$, $bx=-ay$. and we get in a similar manner,

$x=0$, or $\pm a\sqrt{7}$.
$y=0$, or $\mp b\sqrt{7}$.

When $t=-\frac{1}{3}$, $x=0$, or $\pm 3a$.
$y=0$, or $\mp b$.

When $t=-3$, $\quad x=0$, or $\pm a$.
$\quad\quad\quad

$$\frac{\sin^2\alpha}{x}+\frac{\sin^2\beta}{y}+\frac{\sin^2\gamma}{z}=0.$$

By using supplementary angles, the other equations are

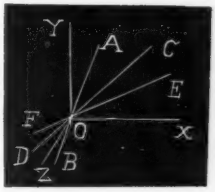
$$\frac{\sin^2\alpha}{x}+\frac{\cos^2\beta}{y}+\frac{\cos^2\gamma}{z}=0,$$

$$-\frac{\cos^2\alpha}{x}+\frac{\sin^2\beta}{y}+\frac{\cos^2\gamma}{z}=0, \text{ and } -\frac{\cos^2\alpha}{x}-\frac{\cos^2\beta}{y}+\frac{\sin^2\gamma}{z}=0.$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let O, the intersection of the three lines, be the origin of rectangular coordinates, and AB, CD, EF the three lines,

$$\left.\begin{matrix}x=az\\y=bz\end{matrix}\right\}\ldots.(1),\quad \left.\begin{matrix}x=cz\\y=dz\end{matrix}\right\}\ldots.(2),\quad \left.\begin{matrix}x=fz\\y=gz\end{matrix}\right\}\ldots.(3),$$



the equations of AB, CD, DE, respectively, and

$$\left.\begin{matrix}x=tz\\y=uz\end{matrix}\right\}\ldots.(4),$$

the equation of the axes of the cone having AO, BO, CO in the surface of one nappe. Then

$$\left.\begin{matrix}x=tz\\y=-uz\end{matrix}\right\}\ldots.(5),\quad \left.\begin{matrix}x=-tz\\y=uz\end{matrix}\right\}\ldots.(6),\quad \left.\begin{matrix}x=-tz\\y=-uz\end{matrix}\right\}\ldots.(7),$$

are the equations of the axes of the remaining three cones.

$$\therefore (x-tz)^2+(y-uz)^2=e^2z^2(1+t^2+u^2)\ldots.(8),$$
$$(x-tz)^2+(y+uz)^2=e_1^2z^2(1+t^2+u^2)\ldots.(9),$$
$$(x+tz)^2+(y-uz)^2=e_2^2z^2(1+t^2+u^2)\ldots.(10),$$
$$(x+tz)^2+(y+uz)^2=e_3^2z^2(1+t^2+u^2)\ldots.(11),$$

are the equations of the cone.

Since (4) must make equal angles with (1), (2), (3), we get at once, if θ is this angle,

$$\cos\theta=\frac{1+at+bu}{\sqrt{[1+a^2+b^2]}\sqrt{[1+t^2+u^2]}}=\frac{1+ct+du}{\sqrt{[1+c^2+d^2]}\sqrt{[1+t^2+u^2]}}$$

$$=\frac{1+ft+gu}{\sqrt{[1+f^2+g^2]}\sqrt{[1+t^2+u^2]}}.$$

$$\therefore \frac{1+at+bu}{\sqrt{[1+a^2+b^2]}}=\frac{1+ct+du}{\sqrt{[1+c^2+d^2]}}=\frac{1+ft+gu}{\sqrt{[1+f^2+g^2]}} \text{ determines } t \text{ and } u.$$

$$e^2=\tan^2\theta=\frac{(t-a)^2+(u-b)^2+(au-bt)^2}{(1+at+bu)^2},\quad e_1^2=\frac{(t-a)^2+(u+b)^2+(au+bt)^2}{(1+at-bu)^2},$$

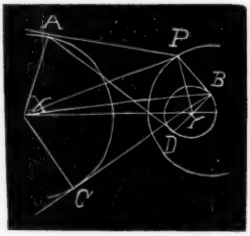
$$e_2^2=\frac{(t+a)^2+(u-b)^2+(au+bt)^2}{(1-at+bu)^2},\quad e_3^2=\frac{(t+a)^2+(u+b)^2+(au-bt)^2}{(1-at-bu)^2}.$$

145. Proposed by FRANK GIFFIN, Graduate Student, State University, Boulder, Col.

If A and B be the points of contact, upon two circles X and Y, of tangents drawn from any point of their circle of similitude, then the tangent from A to Y is equal to the tangent from B to X. [From *Casey's Sequel to Euclid*, Part I., page 144.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let P be any point on the circle of similitude, AP, BP the tangents from P to X and Y, respectively. AD, BC the tangents from A and B to Y and X, respectively.



Let $AX=R$, $BY=r$. Let $AD=a$, $BC=b$, $AP=c$, $BP=d$, $PX=m$, $PY=n$.

$\angle APX=\angle BPY$ since P is on circle of similitude. $\therefore \angle APY=\angle BPX=\theta$.

Also, $c:d=R:r$. $\therefore d=cr/R$....(1).

$m:n=R:r$. $\therefore n=mr/R$....(2).

$$a^2=AY^2-r^2=c^2+n^2-2cn\cos\theta-r^2$$
$$=c^2+m^2r^2/R^2-(2cmr/R)\cos\theta-r^2....(3).$$
$$b^2=RX^2-R^2=d^2+m^2-2dm\cos\theta-R^2=c^2r^2/R^2+m^2-(2cmr/R)\cos\theta-R^2..(4).$$

(3)−(4) gives $R^2(a^2-b^2)=(c^2-m^2+R^2)(R^2-r^2)$.

But $c^2+R^2=m^2$.

$\therefore a^2-b^2=0$. $\therefore a=b$.

CALCULUS.

106. Proposed by M. C. STEVENS, M. A., Professor of Higher Mathematics, Purdue University, Lafayette, Ind.

$$\int_0^\pi \frac{\cos rx\,dx}{1-2a\cos x+a^2}=\frac{\pi a^r}{1-a^2}.$$

[*Williamson's Integral Calculus*, 6th Edition, page 174]

Solution by WILLIAM HOOVER, A.M., Ph.D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

In Todhunter's *Plane Trigonometry*, 3d Edition, Art. 309, we have

$$\cos\alpha+a\cos(\alpha+\beta)+a^2\cos(\alpha+2\beta)+\ldots\ a^{n-1}\cos[\alpha+(n-1)\beta]$$

$$= \frac{\cos\alpha - a\cos(\alpha-\beta)}{1-2a\cos\beta+a^2} \ldots\ldots(1),$$

if n be indefinitely increased. Make $\alpha=0$. Then

$$1+a\cos\beta+a^2\cos 2\beta+\ldots\ldots a^{n-1}\cos[(n-1)\beta] = \frac{1-a\cos\beta}{1-2a\cos\beta+a^2} \ldots\ldots(2).$$

Multiply (2) by 2 and subtract a unit from both members of the resulting equation, then

$$\frac{1-a^2}{1-2a\cos\beta+a^2} = 1+2a\cos\beta+2a^2\cos 2\beta+\ldots\ldots 2a^{n-1}\cos[(n-1)\beta]\ldots\ldots(3).$$

Then, $\int_0^\pi \frac{\cos rx\,dx}{1+2a\cos x+a^2} = \int_0^\pi \frac{\cos rx}{1-a^2}\{1+2a\cos x+2a^2\cos 2x+\ldots\ldots$

$$+2a^{n-1}\cos[(n-1)x]\}dx = \frac{2a^r}{1-a^2}\int_0^\pi \cos^2 rx\,dx = \frac{\pi a^r}{1-a^2},$$

the terms of the series but this one vanishing between the given limits, as may be seen after reducing for a few terms.

Also solved by *G. B. M. ZERR*, *W. W. LANDIS*, *J. SCHEFFER*, and *L. C. PLANT*.

107. Proposed by F. P. MATZ. M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College Mechanicsburg, Pa.

The speed of signaling in submarine telegraph-cable varies as $x^2\log(1/x)$, in which x is the ratio of the radius of the core to that of the covering. Prove that the *maximum speed* is attained when this ratio is $1:\sqrt{e}$.

I. Solution by J. W. YOUNG, Cornell University, Ithaca, N. Y.; COOPER D. SCHMITT, A. M., University of Tennessee, Knoxville, Tenn.; and G. B. M. ZERR, A. M., Ph. D., The Temple College, Philadelphia, Pa.

If $y=x^2\log\left(\frac{1}{x}\right)=-x^2\log x$, $\quad \frac{dy}{dx}=-2x\log x-x=-x(\log x^2+1)$.

$\log x^2+1=0$ gives the maximum.

$\therefore\ \log\frac{1}{x^2}=1.\quad \therefore\ \frac{1}{x^2}=e$, and $x=\frac{1}{\sqrt{e}}$.

II. Solution by H. C. WHITAKER, Ph.D., Manual Training School, Philadelphia, Pa.; W. W. LANDIS, A.M., Dickinson College, Carlisle, Pa.; J. O. MAHONEY, B. E., M. Sc., Central High School, Dallas, Tex.; ALOIS F. KOVARIK, Decorah Institute, Decorah, Ia.; and J. JCHEFFER, A. M., Hagerstown, Md.

If $S=cx^2\log\frac{1}{x}$, $\quad \frac{dS}{dx}=2cx\log\frac{1}{x}-cx$.

Dividing by cx and equating to zero, $\log \frac{1}{x} = \frac{1}{2}$.

Whence, $\frac{1}{x} = \sqrt{e}$, and $x = \frac{1}{\sqrt{e}}$.

MECHANICS.

104. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

From a locomotive and tender standing still on a bridge, the pressure on the bridged is $p_1 = 80$ tons. The track is supposed to be straight and practically horizontal. Had the locomotive and tender been running at the rate of $r = 60$ miles an hour, how many tons would the pressure on the bridge have been?

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Regarding the earth as a perfect sphere and neglecting its rotation, we get for latitude, θ.

$$\text{Centrifugal force} = \frac{Wv^2}{gR\cos\theta} = f,$$

where $W = 80$ tons $= 160000$ pounds, $v = 60$ miles an hour $= 88$ feet per second, $g =$ gravity $= 32.16$ feet, $R =$ radius of earth $= 3956$ miles $= 20887680$ feet.

$$\therefore f = \frac{1239040000}{671747788.8\cos\theta} = \frac{1.8445}{\cos\theta} \text{ pounds.}$$

If the locomotive is running on a great circle, $f = 1.8445$ pounds.

$P =$ pressure $= W - f = 159998.1555$ pounds.

$\therefore P = 79.99908$ tons, or practically 80 tons, the original weight.

105. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $2a$, $2b$ be the diagonals of a rhombus, φ the angle the principal axis at the mid-point of a side makes with the diagonals. Prove $\tan 2\varphi + \frac{3}{5}\tan\beta$, when β is an angle of the rhombus. The principal moments of inertia about this mid-point of the side are $\frac{1}{24}m\{5a^2 + 5b^2 \pm \sqrt{[25(a^2+b^2)^2 - 64a^2b^2]}\}$.

Solution by the PROPOSER.

Let $ABCD$ be the rhombus, side c. EF, FB the axes; $\angle DAB = \angle EFB = \beta$; $\angle\theta =$ the angle the principal axis makes with the side AB at its mid-point F; $\angle\varphi$, the angle the principal axis makes with the diagonal.

$$\text{Then } \Sigma mxy = \rho\sin^2\beta \int_{-\frac{1}{2}c}^{\frac{1}{2}c}\int_0^c (x + y\cos\beta)y\,dx\,dy = \tfrac{1}{3}\rho c^2\sin\beta\cos\beta = \tfrac{1}{3}mc^2\sin\beta\cos\beta.$$

$$\Sigma mx^2 = \rho\sin\beta \int_{-\frac{1}{2}c}^{\frac{1}{2}c}\int_0^c (x + y\cos\beta)^2\,dx\,dy = 1\tfrac{1}{2}m(c^2 + 4c^2\cos^2\beta).$$

$\Sigma my^2=\rho\sin^3\beta\int_{-\frac{1}{2}c}^{\frac{1}{2}c}\int_0^c y^2\,dx\,dy=\frac{1}{3}mc^2\sin^2\beta.$

$\therefore\ \tan 2\theta=\dfrac{4\sin 2\beta}{1+4\cos 2\beta}.$

But $2\theta=\beta+2\varphi$. $\therefore\ \tan 2\varphi=\frac{3}{5}\tan\beta$.

$A\cos^2\theta+\beta\sin^2\theta=\frac{1}{2}m(c^2+4c^2\cos^2\beta)$, $A\sin^2\theta+\beta\cos^2\theta=\frac{1}{3}mc^2\sin^2\beta$.

$\therefore\ A=\frac{1}{24}mc^2[5+\sqrt{(17+8\cos 2\beta)}]$. $B=\frac{1}{24}mc^2[5-\sqrt{(17+8\cos 2\beta)}]$.

$c^2=a^2+b^2$, $\cos\beta=\dfrac{a^2-b^2}{a^2+b^2}$, $\cos 2\beta=\dfrac{(a^2+b^2)^2-8a^2b^2}{(a^2+b^2)^2}$.

$\therefore\ A=\frac{1}{24}m\{5a^2+5b^2+\sqrt{[25(a^2+b^2)-64a^2b^2]}\}$.

$B=\frac{1}{24}m\{5a^2+b^2-\sqrt{[25(a^2+b^2)-64a^2b^2]}\}$.

106. Proposed by J. E. CRAIG, A. B., New Germantown, N. J.

The centers of the two wheels of a bicycle are three feet apart.

(1) If a rider wishes the rear wheel to trace a circle 14 feet in diameter, what must be the diameter of the circle traced by the front wheel?

(2) If the rider weighs 120 pounds, and his center of gravity is 3 feet from the ground, at what angle must he lean to make one revolution of the circle every 3 seconds?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

(1) The line that is tangent to the hind wheel's path passes through the center of the front wheel, caused by the rigid frame work.

$\therefore\ R=\sqrt{(9+49)}=\sqrt{58}=7.615773$ feet.

$2R$=diameter traced by front wheel=15.231546 feet.

(2) Since the man must be in equilibrium we do not use his weight, and we may regard him as moving around in the plane of the track. Suppose he is midway between the wheels.

Let θ be his inclination to the vertical, g=gravity, $f=v^2/r$=centrifugal force. Then $g\sin\theta$=component of gravity perpendicular to man's direction, $f\cos\theta$=component of centrifugal force perpendicular to same direction. When in equilibrium these forces equalize each other.

$\therefore\ g\sin\theta=f\cos\theta=(v^2/r)\cos\theta$.

$\therefore\ \tan\theta=v^2/gr$.

$r=\sqrt{(49+\frac{9}{4})}=7.1589$ feet. $v=14.3178\pi/3$ feet=14.9936 feet per second.

$\therefore\ \tan\theta=\dfrac{(14.9936)^2}{32\frac{1}{6}\times 7.1589}=.97625$.

$\therefore\ \theta=44^\circ\ 18'\ 41''$.

107. Proposed by M. E. GRABER, Student, Heidelberg University, Tiffin, O.

Two particles attracting each other inversely as the square of their distances apart, are constrained to move in straight lines which intersect each other at right angles. How long will it take for the particles to meet and how far does each particle *move?*

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $OA=a$, $OB=b$, $OD=u$, $OC=x$, m=mass of particle at C, n=mass of particle at D; A, B the positions of the particles at the beginning of motion; C, D their positions at any time t; v, v_1 their velocities at C, D. Then the equations of motion are

for C, $\dfrac{d^2x}{dt^2}=\dfrac{nx}{(x^2+u^2)^{\frac{3}{2}}}=F$, since $s=x$,

for D, $\dfrac{d^2u}{dt^2}=\dfrac{mu}{(x^2+u^2)^{\frac{3}{2}}}=f$, since $\sigma=u$. But $v=Ft$, $v_1=ft$.

$$\therefore v_1=\frac{muv}{nx}. \quad \text{Also } \frac{a-x}{v}=\frac{b-u}{v_1}=\frac{(b-u)nx}{muv}.$$

$$\therefore u=\frac{bnx}{am-mx+nx} \ldots(1). \quad \therefore \frac{d^2x}{dt^2}=\frac{n(am-mx+nx)^3}{x^2[a^2n^2+(am-mx+nx)^2]^{\frac{3}{2}}} \ldots(2).$$

From (1), $u=0$ when $x=0$. Therefore, the particles both arrive at O at the same time. Hence C moves over distance a, and D moves over distance b before they meet. The time is found by integrating (if possible) (2) twice.

If $m=n$, $\dfrac{d^2x}{dt^2}=\dfrac{a^3m}{x^2\sqrt{[(a^2+b^2)^3]}}$.

$$\therefore \left(\frac{dx}{dt}\right)^2=\frac{2a^3m}{(a^2+b^2)^{\frac{3}{2}}}\left(\frac{1}{x}-\frac{1}{a}\right)=\frac{2a^2m}{(a^2+b^2)^{\frac{3}{2}}}\cdot\frac{a-x}{x}.$$

$$\therefore t=\frac{(a^2+b^2)^{\frac{3}{4}}}{a\sqrt{2m}}\int_0^a\sqrt{\frac{x}{a-x}}\,dx=\frac{\pi(a^2+b^2)^{\frac{3}{4}}}{2\sqrt{(2m)}}.$$

If $a=b$ and m=unity, $t=\pi(\frac{1}{2}a^2)^{\frac{3}{4}}$.

AVERAGE AND PROBABILITY.

92. Proposed by F. P. MATZ, M.Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

A circular field, radius r, is divided into four *equal* parts, by concentric circles and three concentric rings. From the center of this field are fired *at random*, and with such a velocity as not to produce a range greater than the radius of the field, m=1000 projectiles of the *same* kind. How many projectiles should have fallen into each one of these four equal parts of the field?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The range$=(v^2/g)\sin 2\theta$, where θ=angle of elevation. Greatest range $=v^2/g=r$.

$\therefore\ v^2=gr.\quad \therefore\ (v^2/g)\sin 2\theta=r\sin 2\theta.$

The radii of the three concentric circles are $\frac{1}{2}r\sqrt{3}$, $\frac{1}{2}r\sqrt{2}$, $\frac{1}{2}r$, respectively.

$\therefore\ r\sin 2\theta=\frac{1}{2}rn$, suppose. Solving, we get $\sin\theta=\frac{1}{2}\sqrt{[2\pm\sqrt{(4-n^2)}]}$.

When $n=\sqrt{3}$, $\sin\theta=\frac{1}{2}\sqrt{(2\pm 1)}$. $\therefore\ \theta=\frac{1}{3}\pi$ or $\frac{1}{6}\pi$.

When $n=\sqrt{2}$, $\sin\theta=\frac{1}{2}\sqrt{(2\pm\sqrt{2})}$. $\therefore\ \theta=3\pi/8$ or $\frac{1}{8}\pi$.

When $n=1$, $\sin\theta=\frac{1}{2}\sqrt{(2\pm\sqrt{3})}$. $\therefore\ \theta=5\pi/12$ or $\frac{1}{12}\pi$.

Chance that all fall into outer ring$=p$.

$$\therefore\ p=\frac{r\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}\sin 2\theta d\theta}{r\int_{0}^{\frac{1}{3}\pi}\sin 2\theta d\theta}=\frac{1}{2}.\qquad p_1=\frac{r\int_{\frac{1}{4}\pi}^{\frac{3}{8}\pi}\sin 2\theta d\theta}{r\int_{0}^{\frac{1}{3}\pi}\sin 2\theta d\theta}=\frac{1}{2}\sqrt{2},$$

the chance that all will fall in the two outer rings.

$\therefore\ P=p_1-p=\frac{1}{2}(\sqrt{2}-1)=$the chance that all will fall in the second ring from without.

$$p_2=\frac{r\int_{\frac{1}{12}\pi}^{\frac{5}{12}\pi}\sin 2\theta d\theta}{r\int_{0}^{\frac{1}{3}\pi}\sin 2\theta d\theta}=\frac{1}{2}\sqrt{3},$$ the chance that all will fall in three outer rings.

$\therefore\ P_1=p_2-p_1=\frac{1}{2}(\sqrt{3}-\sqrt{2})=$chance that all fall in third ring from without.

$\therefore\ P_2=1-p_2=\frac{1}{2}(2-\sqrt{3})=$chance that all fall in small circle around the center.

$\therefore$ Number to fall in each space is proportional to these chances, or as $1:(\sqrt{2}-1):(\sqrt{3}-\sqrt{2}):(2-\sqrt{3})$.

$\therefore$ Number in outer ring$=\frac{1}{2}m=500$.

Number in second ring$=m(\sqrt{2}-1)/2=500(\sqrt{2}-1)=207.1065$.

Number in third ring$=m(\sqrt{3}-\sqrt{2})/2=500(\sqrt{3}-\sqrt{2})=158.9185$.

Number in inside circle$=m(2-\sqrt{3})/2=500(2-\sqrt{3})=133.9750$.

93. Proposed by LON C. WALKER, Assistant in Mathematics, Leland Stanford, Jr. University, Palo Alto, Cal.

In Problem 75, required the average area of the circle inscribed in the triangle.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $r=$radius of inscribed circle.

Then $\frac{1}{2}pr=$area$=\frac{1}{2}r[x+y+\sqrt{(x^2+y^2)}]=\frac{1}{2}xy$.

But $x=\dfrac{p^2-2py}{2(p-y)}$. $\therefore\ \frac{1}{2}pr=\dfrac{p(p-2y)y}{4(p-y)}$.

$$\therefore\ r=\frac{(p-2y)y}{2(p-y)},\quad \pi r^2=\frac{\pi y^2(p-2y)^2}{4(p-y)^2}.$$

The limits of y are 0 and $\frac{p}{2+\sqrt{2}}=y'$.

$$\therefore \triangle=\frac{\frac{1}{4}\pi\int_0^{y'}\frac{y^2(p-2y)^2}{(p-y)^2}dy}{\int_0^{y'}dy}=\frac{\pi(2+\sqrt{2})}{4p}\int_0^{y'}\left(4y^2+4py+5p^2+\frac{p^4}{(p-y)^2}-\frac{6p^3}{p-y}\right)dy$$

$$=\frac{\pi p^2}{12}[27-4\sqrt{2}-9(2+\sqrt{2})\log 2].$$

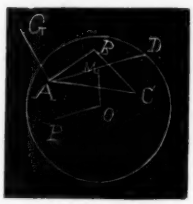
In this solution, as in solution of problem 75, I used the limits of y, 0 and $p/(2+\sqrt{2})$. These limits give all possible variations of size of area. Any other areas are mere repetitions of those included in the above and such a repetition or doubling of areas I believe to be inadmissible.

94. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Three points are taken at random on the surface of the sphere. Find the chance that the triangle thus formed is acute angled.

Solution by the PROPOSER.

Let AD be the diameter of the section of the sphere made by the plane through the three random points A, B, C; M its center; O the center of the sphere; OP a line such that AB is parallel to the plane MOP; p=the chance.



Let $AO=r$, $\angle AOM=\theta$, $\angle GAC=\varphi$, $\angle GAB=\psi$, $\angle MOP=\lambda$, the angle MOP makes with some fixed plane through $OP=\rho$.

An element of the sphere at A is $4\pi r^2\sin\theta d\theta$; at B, $4r^2\sin\theta\sin(\varphi-\psi)\sin\psi\sin\lambda d\psi d\rho$; at C, $4r^2\sin\theta\sin\varphi d\varphi d\lambda$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, $\frac{1}{2}\pi$ and π; of ψ, $\pi-\varphi$ and $\frac{1}{2}\pi$; of λ, 0 and π; of ρ, 0 and 2π.

The three points can be taken $64\pi^3r^6$ ways on the surface of the sphere. Hence

$$p=\frac{1}{64\pi^3r^6}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_{\pi-\phi}^{2\pi}\int_0^{\pi}\int_0^{2\pi}4\pi r^2\sin\theta d\theta.4r^2\sin\theta\sin\varphi d\varphi d\lambda$$

$$\times 4r^2\sin\theta\sin(\varphi-\psi)\sin\psi\sin\lambda d\psi d\rho$$

$$=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_{\pi-\phi}^{\frac{1}{2}\pi}\int_0^{\pi}\sin^3\theta\sin\varphi\sin\psi\sin(\varphi-\psi)\sin\lambda d\theta d\varphi d\psi d\lambda$$

$$=\frac{4}{\pi}\int_0^{\frac{1}{2}\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_{\pi-\phi}^{\frac{1}{2}\pi}\sin^3\theta\sin\varphi\sin\psi\sin(\varphi-\psi)d\theta d\varphi d\psi$$

$$= \frac{1}{\pi}\int_0^{\frac{1}{3}\pi}\int_{\frac{1}{3}\pi}^{\pi} \sin^2\theta(4\sin^2\varphi - 4\sin^4\varphi + \pi\sin\varphi\cos\varphi - 2\varphi\sin\varphi\cos\varphi)d\theta d\varphi$$

$$= \tfrac{1}{2}\int_0^{\frac{1}{3}\pi} \sin^3\theta d\theta = \tfrac{1}{3}.$$

MISCELLANEOUS.

88. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Solve to infinity the series $5\cos\theta = \frac{7\cos 3\theta}{3!} + \frac{9\cos 5\theta}{5!} \ldots\ldots$

Solution by the PROPOSER.

Let the series be $C = \frac{5\cos\theta}{1} + \frac{7\cos 3\theta}{3!} + \frac{9\cos 5\theta}{5!} + \ldots.$

also, $S = \frac{5\sin\theta}{1} + \frac{7\sin 3\theta}{3!} + \frac{9\sin 5\theta}{5!} + \ldots.$

Then $C + Si = \frac{5(\cos\theta + i\sin\theta)}{1} + \frac{7(\cos 3\theta + i\sin 3\theta)}{3!} + \ldots.$

or using a familiar notation,

$$C + Si = \frac{5e^{i\theta}}{1} + \frac{7e^{3i\theta}}{3!} + \frac{9e^{5i\theta}}{5!} + \ldots.$$

which can be written thus:

$$= 4\left(e^{i\theta} + \frac{e^{3i\theta}}{3!} + \frac{e^{5i\theta}}{5!} + \ldots.\right) + \left(e^{i\theta} + \frac{3e^{3i\theta}}{3!} + \frac{5e^{5i\theta}}{5!} + \ldots.\right)$$

$$= 4\left(e^{i\theta} + \frac{e^{3i\theta}}{3!} + \frac{e^{5i\theta}}{5!} + \ldots.\right) + e^{i\theta}\left(1 + \frac{e^{2i\theta}}{2!} + \frac{e^{4i\theta}}{4!} + \ldots.\right)$$

$$= 2\left(e^{e^{i\theta}} - e^{-e^{i\theta}}\right) + \frac{e^{i\theta}}{2}\left(e^{e^{i\theta}} + e^{-e^{i\theta}}\right)$$

But $e^{i\theta} = \cos\theta + i\sin\theta$, so that we have

$$C + Si = 2\left(e^{\cos\theta + i\sin\theta} - e^{-\cos\theta - i\sin\theta}\right) + \frac{e^{i\theta}}{2}\left(e^{\cos\theta + i\sin\theta} + e^{-\cos\theta - i\sin\theta}\right)$$

$$= 2e^{\cos\theta}.e^{i\sin\theta} - 2e^{-\cos\theta}.e^{-i\sin\theta} + \frac{e^{\cos\theta}}{2}.e^{i\theta + i\sin\theta} + \frac{e^{-\cos\theta}}{2}.e^{i\theta - i\sin\theta}$$

$$=2e^{\cos\theta}[\cos(\sin\theta)+i\sin(\sin\theta)]-2e^{-\cos\theta}[\cos(\sin\theta)-i\sin(\sin\theta)]$$

$$+\frac{e^{\cos\theta}}{2}\Big[\cos(\theta+\sin\theta)+i\sin(\theta+\sin\theta)\Big]+\frac{e^{-\cos\theta}}{2}\Big[\cos(\theta-\sin\theta)+i\sin(\theta-\sin\theta)\Big].$$

Now equating the real parts we have,

$$C=2e^{\cos\theta}[\cos(\sin\theta)]-2e^{-\cos\theta}[\cos(\sin\theta)]+\frac{e^{\cos\theta}}{2}[\cos(\theta+\sin\theta)]+\frac{e^{-\cos\theta}}{2}[\cos(\theta+\sin\theta)]$$

which is the required sum.

Also solved by *J. SCHEFFER*, and *G. B. M. ZERR*.

PROBLEMS FOR SOLUTION.

ALGEBRA.

132. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Solve $2^x+3^y=4$; $5^x+6^y=7$.

133. Proposed by HARRY S. VANDIVER, Bala, Montgomery County, Pa.

A theory of Fermat. The sum of two integral fourth powers cannot be an integral square. [Cf. *Chrystal's Algebra*, Vol. II., page 535.]

*** Solutions of these problems should be sent to J. M. Colaw not later than May 10.

GEOMETRY.

161. Proposed by MARCUS BAKER, U. S. Coast and Geodetic Survey Office, Washington, D. C.

A circle, radius r, is inscribed in a triangle ABC. In the angles A, B, and C are inscribed circles each touching two sides and the inscribed circle. There are six such circles. The first group of three have their centers between the incenters and the vertices, and the second group of three does not. Let r_a, r_b, r_c denote the radii of the first group. Then this well known relation holds: $r=\sqrt{(r_ar_b)}+\sqrt{(r_br_c)}+\sqrt{(r_cr_a)}$. Let R_a, R_b, R_c denote the radii of the second group. Then this relation holds:

$$\frac{1}{r}=\frac{1}{\sqrt{(R_aR_b)}}+\frac{1}{\sqrt{(R_bR_c)}}+\frac{1}{\sqrt{(R_cR_a)}}.$$

Required proof.

162. Proposed by J. D. PALMER, Providence, Ky.

Given the distances from the vertices of a triangle, ABC, to the center of the incircle, to construct the triangle.

*** Solutions of these problems should be sent to B. F. Finkel not later than May 10.

CALCULUS.

126. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Find the volume contained between the conical surface whose equation is $z=a-\sqrt{(x^2+y^2)}$, and the planes whose equations are $x=z$ and $x=0$ by the formula $\int\int\int dxdydz$. [*Todhunter's Integral Calculus.*]

127. Proposed by J. A. CALDERHEAD, B.Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

Find the moment of inertia of a parallelogram about an axis perpendicular to its plane and passing through the intersection of its diagonals.

*** Solutions of these problems should be sent to J. M. Colaw not later than May 10.

MECHANICS.

115. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A vessel in the shape of a parallelopiped, filled with water, has in its horizontal bottom a rectangular opening, whose dimensions are a and b, which is shut up by a slider. Supposing this slider to be opened with a uniform motion in the direction of a. To find the depth of the water in the vessel after the time T at the moment when the slider has passed through the space a, a denoting the horizontal section of the water in the vessel.

116. Proposed by C. L. CHILTON, Greensboro, Ala.

Given, the shaft ABC attached at one end by a pivot to the piston-rod of an engine (at A) and the other to the crank of a wheel CDE (at C). The shaft moves through the distance of two feet in one second from A to B and at the same time turns the crank from C to E. The force propelling the shaft along the constrained course from A to B is 5760 pounds. The mass of the rod and wheel and friction being not considered, what would be the kinetic energy of the machine? or the sum of the moment around O, the center of the wheel?

117. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College Mechanicsburg, Pa.

How much lower must *one end* of a heavy uniform chain, wound round the circumference of a perfectly rough vertical wheel, hang than *the other end*, when the chain is on the point of motion?

118. Proposed by M. E. ANDERSON, Minneapolis, Minn.

A closed steel cylinder of length L and diameter D is placed in a horizontal position. The cylinder is filled with water to a depth (a) from the lower side, the space above the water being filled with air at a pressure P_1.

What work will be done against this increasing pressure, and against gravity, by a pump forcing water into this tank until the pressure has increased to P_2? Suppose the level of the water in the tank at the beginning to be the same as that of the reservoir from which the water is pumped.

*** Solutions of these problems should be sent to B. F. Finkel not later than May 10.

AVERAGE AND PROBABILITY.

101. Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

By direct calculation obtain the average distance between to points in the surface of a circle.

102. Proposed by PROFESSOR CAVALLIN.

A random straight line is determined by two points taken at random within a sphere; find the average velocity acquired by a particle in descending the line. [No. 6742, *Educational Times*. Unsolved.]

93. Proposed by LON C. WALKER, Assistant in Mathematics, Leland Stanford, Jr. University, Palo Alto, Cal.

A circle is drawn at random both in magnitude and position, but so as to lie wholly on the surface of a given semi-circle. Show that the chance that a radius drawn at random in the semi-circle will cut the circle is

$$\frac{4}{3\pi-4}\left(1-\frac{1}{\pi}-\frac{2}{\pi}\log 2\right).$$

*** Solutions of these problems should be sent to B. F. Finkel not later than May 10.

MISCELLANEOUS.

102. Proposed by CHARLES C. CROSS, Whaleyville, Va.

Required the least multiple of 17 which when divided by 2, 3, 4, 5, ... 16, leaves, in each case, 1 as a remainder.

103. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics in Boys' High School, Reading, Pa.

Solve, $\log \sin x = \sin \log x$.

104. Proposed by HARRY S. VANDIVER, Bala, Pa.

A Theorem of Fermat. The area of a right angled triangle with commensurable sides cannot be a square number. [Cf. *Chrystal's Algebra*, Vol. II., page 535.]

*** Solutions of these problems should be sent to J, M. Colaw not later than May 10.

NOTES.

Gustav Fock is offering for sale the very valuable mathematical library of Dr. Bruno Christoffel.

Professor M. Cantor of Heidelberg, has been elected a correspondent of the St. Petersburg Academy of Science.

Professor Henry S. White, of Northwestern University, has received leave of absence and will remain abroad until October.

In Italy has just appeared a new mathematical journal issued at Città di Castello by the publisher, S. Lapi, to whom the annual subscription, 12 francs, should be sent. It is a monthly magazine called *Le Matematiche*, under the direction of Prof. C. Alasia with a board of collaborators among whom the English language is represented by G. B. Halsted of Austin, Texas, to whom communications may be sent, which will appear in Italian. On the Editorial Board may also be noted the Russian, Vasiliev, and one of the greatest of living mathematicians, Poincaré. The first number, February, 1901, contains the last thing ever written for publication by the illustrious Hermite, dated January, 1901, on the 14th of which month he died. This number honors THE AMERICAN MATHEMATICAL MONTHLY by reproducing from it in Italian an interesting note. A new and very suggestive department is introduced under the heading, "Subjects for Research."

BOOKS AND PERIODICALS.

Elements of Physics. By Henry A. Rowland, Ph. D., LL. D., Professor of Physics, and Director of the Physical Laboratory in Johns Hopkins University, and Joseph S. Ames, Ph. D., Professor of Physics and Sub-Director of the Physical Laboratory in Johns Hopkins University. 8vo, cloth, xiii+263 pages. Price, $1.00. New York and Chicago: The American Book Company.

This text-book is designed to meet the requirements of high schools and normal schools in the subject of Physics and it is based on the principle that the object of physics is two-fold, viz., (1) to train the student in the powers of observation and accurate description, and (2) to cultivate the habits of exact thought and statement. Great emphasis has been put on those points in the study of the subject which are necessary for the mental training of the student and which will make the more elaborate discussions of the subject simpler when the student comes to them. The reputation of its authors assures the highest authority of statement and great care and thought in its preparation.

B. F. F.

The Common Sense of Commercial Arithmetic. By George Hall, Principal of Petersburg Academy, Petersburg, Va. 8vo. Cloth, xii+187 pages. Price, 60 cents. New York: The Macmillan Co.

This book is designed to give pupils a definite idea of the principles of common sense underlying the subject of Arithmetic common to commercial life. Each subject is clearly and carefully treated, many examples solved in complete detail, and many original problems of practical occurence are proposed for solution. The author's method of presenting percentage is good and his ideas on the subject are sound. B. F. F.

Non-Euclidean Geometry. By Henry Parker Manning, Ph. D., Assistant Professor of Mathematics in Brown University. 8vo, cloth, 94 pages. Boston and Chicago: Ginn & Co.

This is the first attempt in America to present the Non-Euclidean Geometry in a form suitable for use in schools and colleges. The author lays no claim to originality as to subject matter, as much of the work came to the author through the translations of Dr. Halsted. We believe this little book will do much towards popularizing the subject and thus will bring it within the comprehension of teachers of Geometry. In our next issue Dr. Halsted will give an extended review of the work. B. F. F.

The Free Expansion of Gases. Memoirs by Gay-Lussac, Joule ,and Joule and Thompson. Translated and edited by J. S. Ames, Ph. D., Professor of Physics in Johns Hopkins University. 8vo. cloth, 106 pages. Price, $0.75. New York and Chicago: The American Book Co.

This is one of the Series of Memoirs on Physical Subjects, the publication of which was undertaken a few years ago by Harper & Bros. The Series comprises (1) Prismatic and Diffraction Spectra, (2) The Free Expansion of Gases, (3) The Röntgen Rays, (4) The Modern Theory of Solution, and (5) The Laws of Gases. These important memoirs should be in the hands of every teacher of Physics. B. F. F.

Theory of Thought and Knowledge. By Borden P. Bowne, Professor of Philosophy in Boston University. Large 8vo. cloth. xiii+389 pages. Price, $1.50. New York and Chicago: The American Book Co.

This work is written by one of America's ablest thinkers. Professor Bowne deals with the subtilest problems of philosophy in a masterly way. This book consists of two parts; the first part, covering 263 pages, deals with *Theory of Thought* and the second part with the *Theory of Knowledge.* This book is invaluable to the student of Philosophy. B. F. F.

Le Mathematiche. Pure ed Applicate. Periodico mensile di matematiche pure ed applicate, superioai ed elementari, ad uso dell' isturzione media e superore diretto dal Prof. Cristoforo Alasia, con la Collaborazione dei piu illustri Scienziati Italiani e Stranieri.

This monthly journal has just been started, the February number being the first issue. Professor Cristoforo Alasia, the editor, has a large number of very able mathematicians, both native and foreign, to assist him, among whom is Dr. Halsted, who represents America. The MONTHLY wishes this journal the greatest success. B. F. F.

Mathemaitsch-naturwissenschaftliche Mitteilungen be gründet von Dr. O. Böklen, im auftrag des mathematisch-naturwissenschaftlichen Vereins in Württemberg herausgegeben von Dr. A. Schmidt, Dr. A. Haas, und Dr. E. Wölffing.

Inhalt, Januar, 1901. O Böklen, von Dr. E. Wölffing; Vereinsnachrichten; Die nicht-euclidesche Geometrie und die Trigonometrie auf den Flächen von konstantem Krümmungsmass, von Dr. K. Kommerell.

The Mathematical Messenger, Edited and Published by G. H. Harvill, A. M., Athens, Texas.

After a long period of suspension, this journal again makes its appearance. It is published bi-monthly and is devoted chiefly to solutions of problems. The price of the journal is two dollars per year. B. F. F.

Periodico di Mathematica for January and February contains an extensive discussion of the "last theorem of Fermat."

The Mathematical Gazette. Edited by W. J. Greenstreet, M. A. with the coöperation of F. S. McCauley, M. A., D. Sc., and others.

This magazine is published bi-monthly and contains articles, correspondence notes, reviews of books, and problems and solutions. B. F. F.

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NOTE ON POLES AND POLARS.

By GEORGE R. DEAN, C. E., B. Sc., Professor of Mathematics, Missouri School of Mines and Metallurgy, Rolla, Mo.

While the matter here presented is not new to the mathematician, the method appears, from a pedagogical point of view, to have some advantages.

Through the point $O\equiv(x', y')$ a straight line is drawn cutting a conic $ax^2+2hxy+by^2+2gx+2fy+c=0$ at A and B.

Let C be a point in AB such that $OC=\frac{2OA\times OB}{OA+OB}$.

We propose to find the locus of C, when the straight line turns about the point O.

Transferring the origin to (x', y') the new equation is

$$ax^2+2hxy+by^2+2(ax'+hy'+g)+2(hx'+by'+f)+$$
$$ax'^2+2hx'y'+by'^2+2gx'+2fy'+c=0.$$

For the sake of brevity we write this

$$ax^2+2hx'y'+by'^2+2g'x'+2f'y'+c'=0,$$

where $g'=ax'+hy'+g$, $f'=hx'+by'+f$, $c'=ax'^2+2hx'y'+$ etc.

Putting $x=r\cos\theta$, $y=r\sin\theta$, and arranging according to powers of r,

$$(a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta)r^2+2(g'\cos\theta+f'\sin\theta)+c'=0.$$

Then $OC = \frac{2 \text{ product of roots}}{\text{sum of roots}} = \frac{-c'}{g'\cos\theta + f'\sin\theta}$.

Putting $OC = \rho$, we have $g'\rho\cos\theta + f'\rho\sin\theta + c' = 0$.

But $\rho\cos\theta = x$, $\rho\sin\theta = y$; hence $g'x + f'y + c' = 0$.

This straight line is the polar of the point $(0, 0)$ with respect to the conic $ax^2 + 2hxy + by^2 + 2g'x + 2f'y + c' = 0$.

Transferring to the old origin the equation of the polar becomes

$$(ax' + hy' + g)(x - x') + (hx' + by' + f)(y - y') + ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c = 0,$$

which may be written

$$axx' + h(xy' + x'y) + bgy' + g(x + x') + f(y + y') + c = 0.$$

This is the equation of the polar of $x'y'$ with respect to the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

If the point C be fixed then will O describe the polar of C; for, we have on putting $OC = r$, $CA = a$, $CB = b$,

$$r = \frac{2(r + a)(r + b)}{2r + a + b}.$$

Clearing and solving for r, we find $-r = \frac{2ab}{a + b}$, *i. e.* $CO = \frac{2CA \times CB}{CA + CB}$.

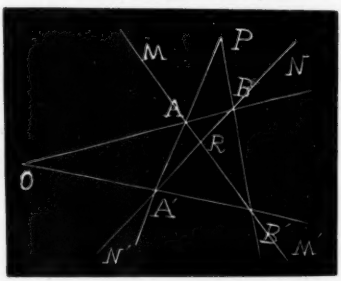
Hence: *If the point P lies on the polar of the point Q, then will Q lie on the polar of P.*

If the line OAB is drawn tangent to the curve the points A and B coincide with C. Hence:

The polar of a point passes through the points of contact of the tangents drawn through the point.

If O be taken on the curve, C will coincide with O, and the polar is the tangent at O.

The polar of a point with respect to a pair of straight lines passes through the intersection of the lines; for, on drawing a line through O and the point of intersection, A and B coincide with C.



To draw the polar of a point with respect to a pair of straight lines: Through O draw two secants cutting the lines at A and B, A' and B', respectively. Join AA', BB' and produce till they meet at P. Join P with common point of given lines R. PR is the polar. For, the polar of AA' and BB' passes through P and cuts AB and $A'B'$ at the same points as the polar with respect to NN' and MM'. The line OR is the polar of P with respect to either of the two pairs of